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Noncommutative $o_\star$(N) and $usp_\star$(2N) Algebras And The Corresponding Gauge Field Theories

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Abstract

The extension of the noncommutative $\mathrm{u}_\star(N)$ Lie algebra to noncommutative orthogonal and symplectic Lie algebras is studied. Using an anti-automorphism of the star-matrix algebra, we show that the $\mathrm{u}_\star(N)$ can consistently be restricted to $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(N)$ algebras that have new mathematical structures. We give explicit fundamental matrix representations of these algebras, through which the formulation for the corresponding noncommutative gauge field theories are obtained. In addition, we present a D-brane configuration with an orientifold which realizes geometrically our algebraic construction, thus embedding the new noncommutative gauge theories in superstring theory in the presence of a constant background magnetic field. Some algebraic generalizations that may have applications in other areas of physics are also discussed.

1 Introduction

Noncommutative (NC) spaces have been shown to arise from string theory [1]. More precisely the world-volume coordinates of Dp-branes living in a constant B-field background turn out to be noncommuting

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}, \quad \mu,\nu = 0,1,\cdots,2n; \quad 2n \leq p+1 \ , \tag{1}$$

where $\theta^{\mu\nu}$ is a $2n \times 2n$ real antisymmetric matrix $\theta^{\mu\nu} = -\theta^{\nu\mu}$, which is a certain function of the background B-field and metric [2]. As a result, the low energy effective theory of the open strings attached to such NC branes becomes a NC gauge theory [2, 3]. The case of NC u(N) gauge theory is well understood, but despite previous attempts [4, 5], the cases of o(N) and usp$(2N)$ have escaped a full understanding. In particular, problems of non-renormalizability have arisen [6] with the previous definition of these theories. In this paper we will introduce a new definition of the noncommutative algebras $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(2N)$ and construct the corresponding gauge field theories. We will also construct the geometry of the D-branes that give rise to these gauge theories. The resulting $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(2N)$ gauge theories look rather different than the previous suggestions.

The main ingredient in the construction at the algebraic level is an anti-automorphism of the noncommutative space. Geometrically, this is related to an orientifold of a new type which had not been considered in brane constructions of gauge theories so far. The new anti-automorphism overcomes certain conceptual problems that were encountered in the previous attempts to construct $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(2N)$ gauge theories.

The star product between two functions $f(x)$ and $g(x)$ over the NC space is defined by the Moyal star product

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial^2}{\partial x_1^\mu \partial x_2^\nu}\right) f(x_1) g(x_2)\bigg|_{x_1=x_2=x} . \tag{2}$$

This associative product defines an algebra $\mathcal{A}$ in the space of functions on NC space. It is clear that the functions that belong to $\mathcal{A}$ are generally complex functions since the star product necessarily introduces the complex number i. Hence, in all of our discussion it will be understood that all non-commutative functions are generically complex; for example, locally, one may think of them as a power series in real coordinates x^μ with complex coefficients. We will define complex conjugation $\bar{f}(x)$ to mean the complex conjugation of the coefficients in the power series.

The star commutator that occurs in Eq.(1) is defined by $[f,g]_\star = f(x) \star g(x) - g(x) \star f(x)$. One may compute some examples of products which will be useful in the discussion below

$$x^\mu \star x^\nu = x^\mu x^\nu + \frac{i}{2}\theta^{\mu\nu}, \tag{3}$$

$$
\begin{aligned}
\left[x^{\mu}, x^{\nu}\right]_{\star} &\equiv x^{\mu} \star x^{\nu} - x^{\nu} \star x^{\mu} = i\theta^{\mu\nu}, &(4)\\
x^{\mu} \star f(x) &= \left(x^{\mu} + \frac{i}{2}\theta^{\mu\nu}\partial_{\nu}\right) f(x), &(5)\\
f(x) \star x^{\mu} &= \left(x^{\mu} - \frac{i}{2}\theta^{\mu\nu}\partial_{\nu}\right) f(x), &(6)\\
x^{\mu} \star x^{\nu} \star x^{\lambda} &= x^{\mu}x^{\nu}x^{\lambda} + \frac{i}{2}\theta^{\mu\nu}x^{\lambda} + \frac{i}{2}\theta^{\mu\lambda}x^{\nu} + \frac{i}{2}\theta^{\nu\lambda}x^{\mu}. &(7)
\end{aligned}
$$

The approach in this paper is rather general and could be used in some other models as well. In particular, it was applied to the classification of various types of higher spin algebras compatible with the dynamics of higher spin gauge fields in AdS_4 [7] and AdS_3 [8]. The higher spin algebras contain subalgebras that correspond to ordinary Yang-Mills symmetries (i.e. spin one) of unitary, orthogonal and symplectic types. Since the higher spin gauge theories are formulated in terms of auxiliary noncommutative spaces with spinor coordinates, the formalism is in many respects analogous to that of the non-commutative Yang-Mills theory (see [9] for reviews and more references on the higher spin gauge theory). Also, examples of algebras that have some relation to those introduced in this paper for the case of noncommutative plane were discussed in other contexts and formalisms, in particular see [10] in the context of the higher spin theories for the case of hyperbolic geometry, and [11] for the toric case.

The paper is organized as follows. In section 2 we present the problem, describe previous attempts, point out some difficulties, and then present an explicit construction of the algebras $\mathrm{o}_{\star}(N)$ and $\mathrm{usp}_{\star}(2N)$ based on an anti-automorphism of the algebra $\mathcal{A}$. The gauge theories follow naturally once the new algebras are defined. In section 3 we present the D-brane geometry that leads to these gauge theories. In section 4 we present a more formal exposition that generalizes the mathematical setup to a broader range of structures that have applications to other problems in physics. In section 5 we give a summary and discuss open problems.

2 Algebraic structure of $\mathbf{o}_{\star}(N)$ and $\mathbf{usp}_{\star}(2N)$

The classical $\mathrm{su}(N)$, $\mathrm{so}(N)$, and $\mathrm{usp}(2N)$ Lie algebras can be defined through their fundamental matrix representations. Namely, one notes that under ordinary matrix commutators the following sets of matrices form Lie algebras: $\mathrm{su}(N)$: $N \times N$ antihermitian traceless matrices $h = -h^{\dagger}$ over complex numbers, $\mathrm{so}(N)$: $N \times N$ antisymmetric matrices $a^t = -a$ over real numbers (where t denotes matrix transposition), and $\mathrm{usp}(2N)$: $2N \times 2N$ antihermitian

matrices that satisfy $s^t = -CsC^{-1}$(equivalently $(Cs)^t = Cs$, is symmetric) where

$$C = \left(\begin{array}{cc} 0 & 1_N \\ -1_N & 0 \end{array} \right). \tag{8}$$

Of course, analytic continuation to various real forms are also possible, but since this is a trivial point in our discussion we will not dwell on it presently. Ordinary gauge theories based on these Lie algebras contain local gauge fields $A_\mu(x)$ that are matrices $(A_\mu(x))^i{}_j$ of the forms h, a, s for su(N), so(N) , usp$(2N)$ respectively. The coordinates x^μ belong to a spacetime manifold M.

In noncommutative gauge theories the spacetime coordinates are replaced by noncommuting coordinates as in Eq.(1). The gauge fields $A_\mu(x)$ belong to $Mat_N \otimes \mathcal{A}$ that is matrices whose matrix elements $(A_\mu(x))^i{}_j$ are noncommutative functions that belong to $\mathcal{A}$.

When the entries of the matrices are elements of $\mathcal{A}$, the product involves not only matrix product but also the star product. This deformation in the product destroys the simple matrix closure of the classical Lie algebras, and a new definition must be introduced to find the sets of matrices $Mat_N \otimes \mathcal{A}$ that close to form the noncommutative versions of the classical Lie algebras. To see the point clearly, suppose we start naively by commuting two so(2) matrices filled with noncommutative functions $f(x)$ and $g(x)$

$$\left(\begin{array}{cc} 0 & f \\ -f & 0 \end{array} \right) \star \left(\begin{array}{cc} 0 & g \\ -g & 0 \end{array} \right) - \left(\begin{array}{cc} 0 & g \\ -g & 0 \end{array} \right) \star \left(\begin{array}{cc} 0 & f \\ -f & 0 \end{array} \right) = \left(\begin{array}{cc} [g,f]_\star & 0 \\ 0 & [g,f]_\star \end{array} \right) \tag{9}$$

If f, g were commutative functions the result would have been zero, corresponding to the closure of ordinary so(2). However, with noncommutative functions f, g the algebra does not close since the result is a different form of matrix. Therefore, we need to find the new sets of matrices over the noncommutative algebra $\mathcal{A}$ that close. We will name the new noncommutative Lie algebras $\mathrm{u}_\star(N;n)$, $\mathrm{o}_\star(N;n)$ and $\mathrm{usp}_\star(2N;n)$, where n is the number of pairs of conjugated coordinates in (1). In the rest of the paper we will skip the label n, however.

The case of $\mathrm{u}_\star(N)$ is well understood and extensively used in the literature. Nevertheless we will include it in our discussion in order to provide a background toward the noncommutative $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(2N)$.

To see what is needed, it is useful to think of the deformation introduced by the star product as being similar to replacing the algebra $\mathcal{A}$ by the algebra of quantum operators or the algebra of matrices. Then one encounters the same closure problem. To find the correct sets of matrices $Mat_N \otimes \mathcal{A}$ that close, the definition of hermitian conjugation or transposition, which entered in the definition of su(N), so(N) or usp$(2N)$, would have to be

extended to the operator or matrix entries as well. Without such a definition the set into which the Lie algebra would close is not specified. The same idea must be applied to the noncommutative algebra $\mathcal{A}$.

Let us consider a map ρ (defined explicitly later) such that, when acting on the elements of $\mathcal{A}$ it has the property

$$\rho\left(\left(f\star g\right)(x)\right)=\left(\rho\left(g\right)\star\rho\left(f\right)\right)(x)\,,\qquad\rho(\rho(f(x)))=f(x)\ ,\tag{10}$$

shared by hermitian conjugation and transposition. The reversal of the orders in the star product is the crucial property. Then, we combine ordinary matrix hermitian conjugation or transposition with the map ρ to define an antiautomorphism for the algebra $Mat_N\otimes\mathcal{A}$. Under the combined operation we demand antihermitian matrices, antisymmetric matrices etc. to define the Lie algebras we are seeking. The main issue is to find an explicit form of the map ρ that works as desired in our context.

2.1 $\mathbf{u}_\star\left(N\right)$

Hermitian conjugation on $\mathcal{A}$ is taken as the standard complex conjugation of a complex function

$$\left(f\left(x\right)\right)^\dagger=\bar{f}\left(x\right)\ ,\tag{11}$$

defined by changing all i into $-i$. This definition is consistent with the star product provided the order of the factors in the star product is reversed

$$\left(f\left(x\right)\star g\left(x\right)\right)^\dagger=\left(g\left(x\right)\right)^\dagger\star\left(f\left(x\right)\right)^\dagger=\bar{g}\left(x\right)\star\bar{f}\left(x\right).\tag{12}$$

This is similar to the hermitian conjugation of the product of quantum operators or matrices. The right hand side indeed gives the ordinary complex conjugate of $f\left(x\right)\star g\left(x\right)$, including the sign change of the i introduced by the star product. This can be proven generally from the definition of the star product in Eq.(2): changing the order of the functions on the right hand side of Eq.(2) is equivalent to changing the sign of $\theta^{\mu\nu}$, which is equivalent to changing the sign of i introduced by the star product. It is perhaps useful to the reader to verify explicitly that the definition works correctly on some explicit products such as those listed in Eqs.(3-7). For example, by using $x^\mu\star x^\nu=x^\mu x^\nu+\frac{i}{2}\theta^{\mu\nu}$ we can evaluate the hermitian conjugation in two ways, first, by complex conjugation of the right hand side, and second, by applying the interchange rule to the star product, thus noting that the result is the same ordinary function of x,θ

$$\left(x^\mu\star x^\nu\right)^\dagger\ =\ x^\mu x^\nu-\frac{i}{2}\theta^{\mu\nu},\tag{13}$$

$$\left(x^\mu\star x^\nu\right)^\dagger\ =\ x^\nu\star x^\mu=x^\mu x^\nu-\frac{i}{2}\theta^{\mu\nu},\tag{14}$$

where we have used $\theta^{\nu\mu} = -\theta^{\mu\nu}$.

To define $\mathrm{u}_\star(N)$ we combine this definition of hermitian conjugation of noncommutative functions with ordinary matrix hermitian conjugation. For the combined hermitian conjugation we require antihermitian matrices. Thus, $\mathrm{u}_\star(N)$ is defined by $N \times N$ matrices whose entries are noncommutative complex functions that satisfy $\left((h(x))^\dagger\right)^i{}_j = -\bar{h}^j{}_i(x)$. The matrices that satisfy these relations have the form

$$u_\star(N): \quad h^i{}_j(x) = \begin{pmatrix} ih_{11}(x) & h_{12}(x) & \cdots & h_{1N}(x) \\ -\overline{h_{12}}(x) & ih_{22}(x) & \cdots & h_{2N}(x) \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{h_{1N(x)}}(x) & -\overline{h_{2N}}(x) & \cdots & ih_{NN}(x) \end{pmatrix} \tag{15}$$

where on the diagonal $ih_{ii}(x)$ are purely imaginary functions, while the off-diagonal $h_{ij}(x)$ are complex functions. (Note that the matrix trace cannot be subtracted since traceless matrices do not close under the combined star-matrix commutation relations. In particular $\mathrm{u}_\star(1)$ is non-Abelian, in contrast to commutative $\mathrm{u}(1)$ which is Abelian.) Such matrices close to form a Lie algebra. Thus, for h_1 and h_2 in the set, the matrix-star commutator results in another h_3 in the set

$$\left([h_1, h_2]_\star\right)^i{}_j \equiv \left(h_1 \star h_2 - h_2 \star h_1\right)^i{}_j = (h_3)^i{}_j\,. \tag{16}$$

Gauge fields $A_\mu(x)$ based on $\mathrm{u}_\star(N)$ are matrices that have the same matrix form as Eq.(15). The corresponding gauge field strength $G_{\mu\nu}$ is also a similar matrix thanks to the closure property

$$(G_{\mu\nu})^i{}_j = \left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star\right)^i{}_j\,. \tag{17}$$

Gauge transformation parameters $(h(x))^i{}_j$ are also similar matrices. Under such gauge transformations $G_{\mu\nu}$ has the usual properties, again thanks to closure

$$\delta A_\mu = \partial_\mu h + [A_\mu, h]_\star\,, \quad \delta G_{\mu\nu} = [G_{\mu\nu}, h]_\star\,. \tag{18}$$

So, in the case of $\mathrm{u}_\star(N)$ the algebraic structure is very similar to that of matrices.

To define an invariant action, the analog of the trace of a matrix is introduced. This must include trace over the matrix as well as over the noncommutative functions. It is well known that the integral over the $2n$ dimensional noncommutative space (for well behaved integrable functions) is the correct definition. Thus, the invariant action for $\mathrm{u}_\star(N)$ gauge theory is

$$S = \int d^{p+1}x\, Tr\left(G_{\mu\nu} G^{\mu\nu}\right), \tag{19}$$

where Tr is the matrix trace.

2.2 $\mathbf{o}_\star(N)$

In [4] a candidate for the map ρ was suggested, and it was called the r-map. It required considering the elements of the algebra $\mathcal{A}$ not simply as functions of spacetime x^μ, but also as functions of the non-commutativity parameter θ, regarded as a variable $f(x;\theta)$. Then the r-map was defined as $(f(x;\theta))^r = f(x;-\theta)$. The authors showed that this definition leads to a Lie algebra which may be called NC o(N), and also presented a D-brane geometry that seemed compatible with a gauge theory based on this Lie algebra. The brane geometry required a B field that was not a constant, but needed to be a step-function in the directions transverse to the orientifold. The main problem in this approach, as anticipated by the authors, is the fact that the θ that appears in the low energy theory is a constant, not a variable. When regarded as a variable, one may expand $Mat(\mathcal{A})$ in a power series in θ and identify the matrix coefficients of the series as independent generators of an infinite dimensional algebra. This requires an infinite set of gauge fields, one for each power of θ; but in string theory there is only one set of gauge fields. The authors speculated that this infinite set could be related to a single set of gauge fields through the Seiberg-Witten map [2]. However, one should anticipate purely on algebraic grounds that unless the infinite set of gauge fields are independent, the corresponding gauge theory is likely to be non-renormalizable. Indeed, signals that this is a problem have already been reported in [6].

We will take here a different approach by defining a new explicit map ρ for a fixed value of θ. In our case the map ρ involves only the spacetime coordinates x^μ. To define $\mathrm{o}_\star(N)$ we introduce the analog of transposition for the non-commutative functions. We will introduce two such transpositions denoted by t_1 and t_2 (candidates for ρ) as follows

$$(f(x_1,x_2))^{t_1} = f(x_1,-x_2) \quad or \quad (f(x_1,x_2))^{t_2} = f(x_2,x_1). \tag{20}$$

These two definitions are algebraically equivalent up to a redefinition of the coordinates, $x_\pm = (x_1 \pm x_2)/\sqrt{2}$. Here we have assumed a 2-dimensional noncommutative space, $[x_1,x_2]_\star = i\theta$, for simplicity of the presentation, but the generalization to higher dimensions is obvious, for example by promoting the pairs of noncommuting coordinates to vectors $\vec{x}_1,\vec{x}_2$. The functions f can depend on additional commuting coordinates. These are assumed to be present, but they will be suppressed for the sake of a simpler presentation. These transpositions satisfy the following property under star products

$$(f(x)\star g(x))^{t_1} = (g(x))^{t_1}\star(f(x))^{t_1} = (f\star g)(x_1,-x_2), \tag{21}$$

$$(f(x)\star g(x))^{t_2} = (g(x))^{t_2}\star(f(x))^{t_2} = (f\star g)(x_2,x_1). \tag{22}$$

That is, the order of the functions in a product is reversed before applying the transposition on the individual functions. As noted before, the noncommutative functions are generically

complex, but the complex number i does not transform under the transpositions t_1 or t_2. This again is similar to the rules obeyed by matrices under matrix transposition.

In fact, one can make the analogy to matrix transposition rather explicit by recalling the matrix representation for the functions on the noncommutative torus (using the clock-shift matrices e^{ix_1}=clock, e^{ix_2}=shift, see e.g. [12]) or the noncommutative plane (e.g. see [13]). In this point of view the map t_1 is nothing but the usual transposition in the matrix representation: the change of basis in the matrix representation is equivalent to the coordinate reflection on the noncommutative space. The complex number i, or the sign of θ do not change sign under matrix transposition.

To get familiar with the t_1 operation we record a few examples by making use of the computations listed in Eq.(3-7)

$$\begin{aligned}
(x_1 \star x_2)^{t_1} &= \left(x_1 x_2 + \frac{i}{2}\theta\right)^{t_1} = -x_1 x_2 + \frac{i}{2}\theta && (23)\\
&= (x_2)^{t_1} \star (x_1)^{t_1} = -x_2 \star x_1 = -x_1 x_2 + \frac{i}{2}\theta && (24)\\
(x_2 \star x_1)^{t_1} &= -(x_1)^{t_1} \star (x_2)^{t_2} = -x_1 x_2 - \frac{i}{2}\theta && (25)\\
([x^\mu, x^\nu]_\star)^{t_1} &= i\theta^{\mu\nu} && (26)\\
(x_1 \star f(x_1, x_2))^{t_1} &= \left(\left(x^1 + \frac{i\theta}{2}\frac{\partial}{\partial x^2}\right) f(x_1, x_2)\right)^{t_1} = \left(x^1 - \frac{i\theta}{2}\frac{\partial}{\partial x^2}\right) f(x_1, -x_2) && (27)\\
&= f(x_1, -x_2) \star x_1 = (f(x_1, x_2))^{t_1} \star (x_1)^{t_1} \,. && (28)
\end{aligned}$$

Similar exercises using t_2 are left to the reader.

We are now ready to define the set of matrices that form $\mathrm{o}_\star(N)$. We combine the ordinary transposition of matrices, $(a^t)_{ij} = a_{ji}$, with the transpositions t_1 or t_2 applied on the noncommutative functions, and denote the combined operation with the letter T. Then $\mathrm{o}_\star(N)$ is given by the antisymmetric matrices under the combined operation. Thus, for the cases t_1 or t_2 we demand that a matrix $a \in \mathrm{o}_\star(N)$ satisfies

$$\begin{aligned}
t_1 &: \quad \left((a(x_1, x_2))^T\right)_{ij} \equiv \left(a^t\right)_{ij}(x_1, -x_2) = a_{ji}(x_1, -x_2) = -a_{ij}(x_1, x_2)\,, && (29)\\
t_2 &: \quad \left((a(x_1, x_2))^T\right)_{ij} \equiv \left(a^t\right)_{ij}(x_2, x_1) = a_{ji}(x_2, x_1) = -a_{ij}(x_1, x_2)\,. && (30)
\end{aligned}$$

More explicitly, for the case of t_1 the parameters of $\mathrm{o}_\star(N)$ must have the form

$$o_\star(N)_{t_1} : a_{ij}(x_1, x_2) = \begin{pmatrix} a_{11}(x_1, x_2) & a_{12}(x_1, x_2) & \cdots & a_{1N}(x_1, x_2) \\ -a_{12}(x_1, -x_2) & a_{22}(x_1, x_2) & \cdots & a_{2N}(x_1, x_2) \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1N}(x_1, -x_2) & -a_{2N}(x_1, -x_2) & \cdots & a_{NN}(x_1, x_2) \end{pmatrix} \qquad (31)$$

That is, the diagonal elements are odd functions under reflections of x_2, i.e. $a_{ii}(x_1, x_2) = -a_{ii}(x_1, -x_2)$, while the lower off-diagonal elements are related to the upper off-diagonal elements by reflections of x_2. Note that the upper off-diagonal elements do not have any symmetry properties under the reflections. Under t_2 one can define a similar $\mathrm{o}_\star(N)_{t_2}$ whose structure differs from the one above only by replacing the reflections of x_2 by the interchange $x_1 \longleftrightarrow x_2$.

Such matrices close under the matrix-star commutation relations. Thus, for a,$b \in \mathrm{o}_\star(N)$, the matrix-star commutator results in another matrix $c \in \mathrm{o}_\star(N)$

$$([a, b]_\star)_{ij} \equiv (a \star b - b \star a)_{\ ij} = (c)_{\ ij} \tag{32}$$

In particular note that $\mathrm{o}_\star(1)$ exists non-trivially. It consists of functions that are antisymmetric under reflections of x_2 or under the interchange $x_1 \longleftrightarrow x_2$

$$o_\star(1)_{t_1} \quad : \qquad a(x_1, x_2) = -a(x_1, -x_2), \tag{33}$$

$$o_\star(1)_{t_2} \quad : \qquad a(x_1, x_2) = -a(x_2, x_1). \tag{34}$$

Such *odd* functions close under star commutators.

All entries in $\mathrm{o}_\star(N)_{t_{1,2}}$ are generically complex functions, however it is possible to restrict the dependence on the complex number i as follows. In addition to the transposition operation T of Eqs.(29,30), we also impose the hermiticity condition as in the $u_\star(N)$ case, i.e. $a^\dagger = -a$. These two conditions are compatible with each other. This shows that our $\mathrm{o}_\star(N)_{t_{1,2}}$ form a subalgebra of the $\mathrm{u}_\star(N)$. The matrices a that satisfy both conditions are of the form (31) such that the upper off diagonal elements are of the form $a_{i<j} = a^+_{i<j} + is^-_{i<j}$ where $a^+_{i<j}$ is symmetric and $s^-_{i<j}$ is antisymmetric under reflections, $a^+_{i<j}(x_1, -x_2) = a^+_{i<j}(x_1, x_2)$, $s^-_{i<j}(x_1, -x_2) = -s^-_{i<j}(x_1, x_2)$, while the diagonal elements $a_{ii} = ia^-_{ii}$ are purely imaginary as well as antisymmetric under reflections. Examining only the a^+_{ij} part of the matrix a, we see that it is real and has the same form of $\mathrm{so}(N)$ matrices; however these by themselves do not close under the combined star and matrix products. Closure requires also the imaginary parameters $is^-_{i<j}$ and is^-_{ii}. Hence, denoting an antihermitian a with the symbol a_h, we may write

$$a_h = a^+ + is^-, \tag{35}$$

where a^+ is even under reflections and is an antisymmetric matrix, while s^- is odd under reflections and is a symmetric matrix. Both a^+ and s^- are real. Note that the number of degrees of freedom in these parameters is fewer than those in $\mathrm{u}_\star(N)$ because for $\mathrm{u}_\star(N)$ the parameters are not restricted by the reflection conditions.

Note also that the usual so(N) is a subalgebra of $\mathrm{o}_\star(N)_{t_1}$ when all entries are functions of only x_1, i.e. $a_{ij}(x_1)$, since then all s_{ij}^- vanish, and the remaining matrix is a^+ real. For such matrices the star-matrix commutator collapses to ordinary matrix commutator, and they obviously form the so(N) Lie algebra. Similarly so(N) is a subalgebra of $\mathrm{o}_\star(N)_{t_2}$ when all entries are functions of only $x_1+x_2 : a_{ij}(x_1+x_2)$. This implies in particular that the usual global $so(N)$ symmetry with the $x-$independent parameters is the subalgebra of $\mathrm{o}_\star(N)_{t_{i,2}}$.

In the following we will also need matrices $s_{ij}(x_1,x_2)$ that are symmetric under the T operation, that is

$$t_1 \quad : \qquad \left((s(x_1,x_2))^T\right)_{ij} \equiv \left(s^t\right)_{ij}(x_1,-x_2) = s_{ji}(x_1,-x_2) = s_{ij}(x_1,x_2), \tag{36}$$

$$t_2 \quad : \qquad \left((s(x_1,x_2))^T\right)_{ij} \equiv \left(s^t\right)_{ij}(x_2,x_1) = s_{ji}(x_2,x_1) = s_{ij}(x_1,x_2). \tag{37}$$

Explicitly, such matrices have the form (using t_1)

$$t_1 : s_{ij}(x_1,x_2) = \begin{pmatrix} s_{11}(x_1,x_2) & s_{12}(x_1,x_2) & \cdots & s_{1N}(x_1,x_2) \\ s_{12}(x_1,-x_2) & s_{22}(x_1,x_2) & \cdots & s_{2N}(x_1,x_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_{1N}(x_1,-x_2) & s_{2N}(x_1,-x_2) & \cdots & s_{NN}(x_1,x_2) \end{pmatrix} \tag{38}$$

Generally the entries are complex. If we impose the antihermiticity conditions, then the diagonal is purely imaginary and even $s_{ii} = is^+(x_1,x_2)$, while the upper off-diagonal has the form $s_{i<j} = a_{i<j}^-(x_1,x_2) + is_{i<j}^+(x_1,x_2)$, where $a_{i<j}^-(x_1,x_2)$ are odd and $s_{i<j}^+(x_1,x_2)$ are even under reflections. That is, the matrix elements of s have opposite reflection properties to those of the a matrix discussed above. Hence, denoting an antihermitian s with the symbol s_h, we may write

$$s_h = a^- + is^+ \tag{39}$$

where a^- is odd under reflections and is an antisymmetric matrix, while s^+ is even under reflections and is a symmetric matrix. Both a^- and s^+ are real. Similar structures arise if we consider t_2 instead of t_1. It is interesting to note schematically the matrix-star commutation properties of these types of matrices

$$[a,a']_\star \sim a'', \quad [a,s]_\star \sim s', \quad [s,s']_\star \sim a. \tag{40}$$

This closure property applies to the general complex a, s as well as to the hermitian subsets a_h, s_h.

We now consider matrix gauge fields $A_\mu\left(x^0, x_{1\alpha}, x_{2\alpha}, y^I\right)$, $\alpha = 1,2,...,n$, which depend on $\left(x^0, x_{1\alpha}, x_{2\alpha}, y^I\right)$, where x^0 is the time coordinate, y^I denotes commuting coordinates and $\vec{x}_1, \vec{x}_2$ are pairs of non-commuting coordinates on which we apply the $t_{1,2}$ operations.

t_1 could be applied to some components of the vectors, while t_2 could be applied on the remaining components. For definiteness consider only t_1. The spacetime index takes the values $\mu = 0, 1\alpha, 2\alpha, I$. The gauge parameters are matrices $a_h\left(x^0, \vec{x}_1, \vec{x}_2, y\right)$ of the form (35). Note that, in principle, one can also apply reflections like $t_{1,2}$ to the commuting coordinates y^I. In particular, theories of this type will result in the limit $\theta \to 0$ for some of the non-commutative coordinates.

The 1-forms A_μ are matrices which transform under $\mathrm{o}_\star(N)$ according to the rule

$$\delta A_\mu = \partial_\mu a_h + A_\mu \star a_h - a_h \star A_\mu. \tag{41}$$

Let us write these transformations explicitly for the various components $\mu = (0, 1\alpha, I) \equiv m$, and $\mu = 2\alpha$

$$\begin{aligned} \delta A_m &= \partial_m a_h + A_m \star a_h - a_h \star A_m, && (42)\\ \delta A_{2\alpha} &= \partial_{2\alpha} a_h + A_{2\alpha} \star a_h - a_h \star A_{2\alpha}. && (43) \end{aligned}$$

The gauge fields, A_0, $A_{1\alpha}$, A_I have the same matrix form as a_h as in (35), as usual. However, taking into account the reflection $\vec{x}_2 \to -\vec{x}_2$ we must conclude that the gauge field $A_{2\alpha}$ cannot be of that form. This is seen by examining the term $\partial_{2\alpha} a$ which is not of the form a_h after applying the derivative, but rather it has the form s_h as in (39). For consistency, we must also demand that $A_{2\alpha}$ is a matrix of the form s_h as in (39). Then the remainder of the transformation $\delta A_{2\alpha}$ is consistent according to Eq.(40). These observations are compatible with the coupling of the gauge field to a current. For example, consider a particle coupled to the field A_μ, for which the Lagrangian contains the term $\dot{x}^\mu A_\mu(x)$. The symmetry of the Lagrangian under the reflections $\vec{x}_2 \to -\vec{x}_2$ requires opposite reflection properties from $A_{2\alpha}$ versus A_0, $A_{1\alpha}$, A_I. When we consider a string coupled to a stack of branes in the next section the same observations will be valid.

Next, consider the gauge field strength $G_{\mu\nu}\left(x^0, \vec{x}_1, \vec{x}_2, y\right) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star$ and examine its different components

$$\begin{aligned} G_{mn} &= \partial_m A_n - \partial_n A_m + [A_m, A_n]_\star && (44)\\ G_{m,2\alpha} &= \partial_m A_{2\alpha} - \partial_{2\alpha} A_m + [A_m, A_{2\alpha}]_\star && (45)\\ G_{2\beta,2\alpha} &= \partial_{2\beta} A_{2\alpha} - \partial_{2\alpha} A_{2\beta} + [A_{2\beta}, A_{2\alpha}]_\star && (46) \end{aligned}$$

We see from the derivative terms and from Eq.(40) that G_{mn} and $G_{2\beta,2\alpha}$ have the form of a_h, while $G_{m,2\alpha}$ has the form of s_h. However, all components transform under the same rule under $\mathrm{o}_\star(N)$ gauge transformations (41)

$$\delta G_{\mu\nu} = G_{\mu\nu} \star a_h - a_h \star G_{\mu\nu}. \tag{47}$$

Therefore, an invariant action for $o_\star(N)$ gauge theory is

$$S = \int dx^0 d^{p-2n} y \int d^{2n} x \, Tr \left(G_{\mu\nu} G^{\mu\nu} \right), \tag{48}$$

where Tr is the matrix trace.

In particular, there is an $o_\star(1)$ gauge theory. The $o_\star(1)$ gauge field has fewer degrees of freedom than the $u_\star(1)$ gauge field because A_μ can be taken as purely imaginary functions that are odd/even under reflections (indicated by $\pm$ superscripts) $iA_m^-(x^0, x_1, x_2, y)$ and $iA_{2\alpha}^+(x^0, x_1, x_2, y)$. By contrast the $u_\star(1)$ gauge fields A_μ are purely imaginary functions that have no definite reflection symmetry properties.

Let us stress that the action (48) is invariant because both the integration measure and the combination of the metric tensors $g^{\mu\nu}$ used to contract the indices of the noncommutative field strength $G_{\mu\nu} G^{\mu\nu}$ are invariant under the reflections $t_{1,2}$ when $\theta^{\mu\nu}$ is block diagonal. More generally, one may examine the invariance of the action when $\theta^{\mu\nu}$ is not block diagonal. Under SO$(2n)$ transformations in the noncommutative directions, the form of the action does not change provided the metric $g^{\mu\nu}$ in $2n$ dimensions is rotationally invariant. But through the SO$(2n)$ rotations of the coordinates, $x^\mu \to \Lambda^{\mu\nu} x_\nu$, the constant $\theta^{\mu\nu}$ gets transformed into a new value $\left(\Lambda\theta\Lambda^T\right)^{\mu\nu}$. It is always possible to choose Λ such that $\Lambda\theta\Lambda^T$ is block diagonal. Thus, beginning with an action with general $\theta^{\mu\nu}$, and a rotationally invariant metric $g^{\mu\nu}$, one can always transform the action to the coordinate basis in which $\theta^{\mu\nu}$ has a block diagonal form. This is the basis in which we had defined the $t_{1,2}$ previously. Now we see that we can define more general $t_{1,2}$ for general $\theta^{\mu\nu}$ by conjugating the previous ones with the SO$(2n)$ rotation Λ. This argument shows that for general $\theta^{\mu\nu}$, there exist some $t_{1,2}$ that play the same role as before. Therefore, if the metric has rotation symmetry, then the action has the other desired symmetries for general $\theta^{\mu\nu}$, not only for block diagonal forms.

Finally, we point out properties of the model in the $\theta \to 0$ limit. In this "classical limit" of the $o_\star(N)$ (similarly $usp_\star(2N)$) gauge theory, the resulting commutative field theory has a richer structure compared to the usual pure Yang-Mills theory. The outcome depends on which quantities are held fixed as the limit is taken. First, one may consider a straightforward $\theta \to 0$ limit in which the gauge potentials $(A_m, A_{2\alpha})$ continue to have the forms (a_h, s_h) of Eqs.(35,39) respectively. Then different polarizations behave differently under the transposition T even in the commutative limit. However, since the imaginary parts of (a_h, s_h) would no longer be needed for the closure of the algebra, it is also possible to consider a second $\theta \to 0$ limit in which the normalization of the imaginary part is rescaled by a power of θ before taking the limit. Then the resulting theory has only gauge potentials $\left(A_m^+, A_{2\alpha}^-\right)$ that are real antisymmetric matrices (in the adjoint representation of standard so(N)), but still have respectively definite $(even, odd)$ properties under reflections t_1, as indicated by the

$\pm$ labels. This is still different than the usual Yang-Mills theory. Next we notice that under dimensional compactification, in which all dependence on $x_{2\alpha}$ is eliminated, the theory reduces to a familiar Yang-Mills type theory, in which the $N \times N$ antisymmetric matrices in the remaining fewer dimensions $A_m^+\left(x^0, x_{1\alpha}, y^I\right)$ become the standard so(N) Yang-Mills gauge fields in the adjoint representation, while the $A_{2\alpha}^-\left(x^0, x_{1\alpha}, y^I\right)$ become scalar fields that are also in the adjoint representation (the $\pm$ labels not needed anymore). It is possible to achieve this reduction by considering yet a third $\theta \to 0$ limit, in which the $x_{2\alpha}$ dependence of the functions is rescaled by θ in the form $A_m\left(x^0, x_1, \theta x_2, y\right)$ and $A_{2\alpha}\left(x^0, x_1, \theta x_2, y\right)$, so that $x_{2\alpha}$ completely disappear from the functions when θ vanishes. Finally, one may modify the last case with a $\theta \to 0$ limit that keeps both the real and imaginary parts of the scalars $A_{2\alpha}\left(x^0, x_1, y\right) = A_{2\alpha}^-\left(x^0, x_1, y\right) + iA_{2\alpha}^+\left(x^0, x_1, y\right)$ in which $A_{2\alpha}^+$ are symmetric $N \times N$ matrices.

2.3 usp$_\star(2N)$

To define usp$_\star(2N)$ we use the $t_{1,2}$ transpositions defined in the previous subsection and combine them with matrix transposition. Then we define matrices that satisfy the condition $S^T = -CSC^{-1}$, and $V^T = CVC^{-1}$, where the operation T is defined in Eqs.(29,30) and the matrix C is given in Eq.(8). We further require these matrices to be antihermitian. Then they form subsets of the u$_\star(2N)$ matrices. The matrices that satisfy these condition have the form

$$S = \begin{pmatrix} h & s \\ -s^\dagger & -h^T \end{pmatrix}, \quad V = \begin{pmatrix} h' & a \\ -a^\dagger & (h')^T \end{pmatrix}. \tag{49}$$

Here $\left(h\left(x_1, x_2\right)\right)^i{}_j$ or $\left(h'\left(x_1, x_2\right)\right)^i{}_j$ are antihermitian $N \times N$ matrices, as in Eq.(15), and $\left(h\left(x_1, x_2\right)\right)^T$ or $\left(h'\left(x_1, x_2\right)\right)^T$ are their transpose given by transposition of the matrix combined with the t_1 or t_2 operation applied on the functions of $\left(x_1, x_2\right)$. Similarly, $s\left(x_1, x_2\right)$ and $a\left(x_1, x_2\right)$ are independent $N \times N$ matrices that are symmetric/antisymmetric under the transposition T, so they have the forms of s, a as in Eqs.(38,31) respectively, but they are not required to be antihermitian (i.e. do not impose the additional conditions discussed following those equations). Matrices of the type (49), close under the star-matrix commutation rules and form the usp$_\star(2N)$ Lie algebra

$$\left(\left[S_1, S_2\right]_\star\right)_{IJ} \equiv \left(S_1 \star S_2 - S_2 \star S_1\right)_{IJ} = \left(S_3\right)_{IJ}. \tag{50}$$

Evidently, this is a subalgebra of u$_\star(2N)$. Furthermore, the matrices S, V close as follows

$$\left[S, S'\right]_\star \sim S'', \quad \left[S, V\right]_\star \sim V, \quad \left[V, V'\right]_\star \sim S. \tag{51}$$

One can now proceed to construct usp$_\star(2N)$ gauge theory by introducing gauge fields $A_\mu\left(x_1, x_2\right)$. As in the previous section, there is a difference between A_0, A_I, A_{1i} and A_{2i}. The

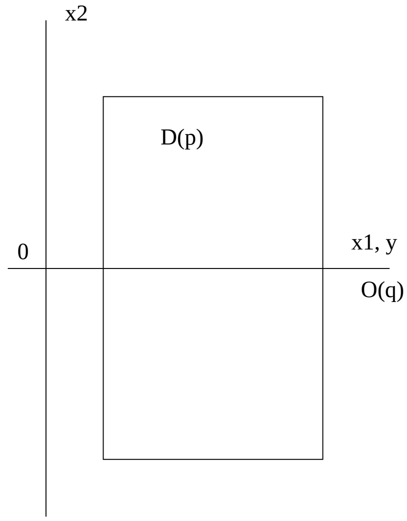

Figure 1: D_p brane and orientifold O_q plane

A_0, A_I, A_{1i} are matrices of the form S as in Eq.(49) while the A_{2i} have the form of V as in Eq.(49).

The construction of the field strength, its consistent properties, and the invariant action are then easily understood, as in the previous subsection, as consequences of the closure among such matrices, Eq.(51).

3 Geometric construction with D-branes

$\mathrm{u}_\star(N)$ gauge theories can be understood as a consequence of open string theory in the presence of a constant background magnetic field with the string boundaries ending on D-branes. When the branes collapse on top of each other, and in the large background field limit, the $\mathrm{u}_\star(N)$ gauge theory captures the physics of the strings and D-branes system [2, 3]. What is the analog geometric construction that corresponds to the $\mathrm{o}_\star(N)$ and the $\mathrm{usp}_\star(2N)$ gauge theories we introduced in the previous section?

The answer can be found through brane configurations involving orientifolds, chosen so that they realize the transpositions t_1 (or t_2). In addition we also need our brane system to be supersymmetric. In order to find a $p+1$ dimensional theory (including time X^0), let us consider D_p-branes which are along $1\cdots p$ spacelike directions. Let $(p-2n)$ of them be spacelike commutative coordinates y^I and $2n$ of them be our spacelike noncommutative coordinates $(x_{1\alpha}, x_{2\alpha})$. More clearly, in our previous notations,

$$
\begin{aligned}
X^\mu, \quad & \mu = 0 \ \text{ is the time coordinate,}\\
& \mu = 1\alpha, \text{with odd } \ 1\alpha = 1, 3, \cdots, 2n-1, \ \text{ are the noncommuting } x_{1\alpha} \text{ coordinates,}\\
& \mu = 2\alpha, \text{ with even } 2\alpha = 2, 4, \cdots, 2n, \ \text{ are the noncommuting } x_{2\alpha} \text{ coordinates,}\\
& \mu = 2n+1, \cdots, p \ \text{ are the commuting } y^I \text{ coordinates,}
\end{aligned}
$$

Then we choose our O$_q$-plane so that it has common coordinates with our D$_p$-brane in $y^I, x_{1\alpha}$ directions, while $x_{2\alpha}$ are transverse to the O$_q$-plane as in figure (1).

Also, to satisfy the supersymmetry conditions (before turning on the B field), one may start from a configuration with D$_p$ branes and an O$_{p\pm 4k}$ plane. Then one may compactify one or more of the extra dimensions on the D-brane and apply T-duality to obtain additional supersymmetric configurations. Thus, we may start with D$_p$+O$_{p-4}$. Compactifying one dimension and applying T-duality gives D$_{p-1}$+ O$_{p-3}$; compactifying two dimensions and applying T-duality gives D$_{p-2}$+O$_{p-2}$; compactifying three dimensions and applying T-duality gives D$_{p-3}$+O$_{p-1}$. Similarly, one could start with the systems D$_p$+O$_{p+4}$, etc. and apply the same procedure. We may shift the value of p so that we always have a D$_p$ brane plus an associated O$_q$ obtained as above. We also note that for our scheme we need $2n \leq p \leq 9$ and $q \geq p-n$. This reasoning fixes the possible number of dimensions and configurations of the D$_p$+O$_q$ of interest for our work to the following

$$
D_p + O_{p-2n+4k}, \quad k = 1, 2. \tag{52}
$$

Such brane configurations will preserve 1/4 of the 32 supersymmetries of the type II theory. However, as we will show, the same amount of supersymmetry remains after turning on the B-field; we will momentarily come to this point. Supersymmetry guarantees the stability of the system from the point of view of the complete string-brane theory, and insures that the field theory limit makes sense as part of a finite theory.

Let us consider the D$_p$+O$_q$ configurations of interest in more detail, in order of increasing n. For $n = 1$, the configuration (D$_p$+O$_{p+2}$) is described by

$$
\begin{array}{rccccccl}
N \ \ D_p - \text{branes}: & 0 & 1\ 2 & 3\cdots p & - & - & - & , \quad \text{with } B_{12}\\
O_{p+2} - \text{plane}: & 0 & 1\, - & 3\cdots p & p+1 & p+2 & p+3 & .
\end{array} \tag{53}
$$

where a "$-$" indicates that the D-brane or O-plane does not occupy the corresponding dimensions. The reflections occur for the dimensions not occupied by the O-plane, i.e. dimensions marked by "$-$" for the O-plane (dimension $\mu = 2$ in the present example). Therefore, the B_{12} field is taken with one of its indices along this dimension. This configuration realizes a $p+1$ dimensional gauge theory on the D$_p$ brane worldvolume. It has up to two noncommuting coordinates x_1, x_2 (which become commuting if $B_{12} = 0$), a commuting time coordinate, and $p-2$ commuting y^I coordinates for $2 \leq p \leq 9$. We will argue below that this leads

to the gauge group $o_\star(N;1)$ for N odd or even (or $usp_\star(N;1)$ with N=even), where $n=1$ indicates the number of noncommuting pairs.

In the case of $n=1$ and $p=2$ we can also consider (D_2+O_8) in a similar way to (D_2+O_4) by adding 4 more dimensions to the O-plane in Eq.(53). These additional dimensions are not occupied by the D brane, so they do not show up in the low energy gauge theory.

For $n=2$, the configuration (D_p+O_p) is described by

$$\begin{array}{lcccccccl} N\ \ D_p-\text{branes}: & 0 & 1\,2 & 3\,4 & 5\cdots p & - & - & , & \text{with } B_{12}, B_{34} \\ O_p-\text{plane}: & 0 & 1- & 3- & 5\cdots p & p+1 & p+2 & . & \end{array} \tag{54}$$

The reflections occur for the dimensions labelled by $\mu = 2\alpha = 2,4$. With B_{12}, B_{34} there are up to 4 non-commuting coordinates, a commuting time coordinate, and $p-4$ commuting y^I coordinates for $4 \le p \le 9$.

For $n=3$, the configuration (D_p+O_{p-2}) is described by

$$\begin{array}{lcccccccl} N\ \ D_p-\text{branes}: & 0 & 1\,2 & 3\,4 & 5\,6 & 7\cdots p & - & , & \text{with } B_{12}, B_{34}, B_{56} \\ O_{p-2}-\text{plane}: & 0 & 1- & 3- & 5- & 7\cdots p & p+1. & & \end{array} \tag{55}$$

The reflections occur for the dimensions labelled by $\mu = 2\alpha = 2,4,6$. With B_{12}, B_{34}, B_{56} there are up to 6 non-commuting coordinates, a commuting time coordinate, and $p-6$ commuting y^I coordinates for $6 \le p \le 9$.

For $n=4$, the configuration (D_p+O_{p-4}) is described by

$$\begin{array}{lcccccccl} N\ \ D_p-\text{branes}: & 0 & 1\,2 & 3\,4 & 5\,6 & 7\,8 & p, & \text{with } B_{12}, B_{34}, B_{56}, B_{78} \\ O_{p-4}-\text{plane}: & 0 & 1- & 3- & 5- & 7- & p. & \end{array} \tag{56}$$

The reflections occur for the dimensions labelled by $\mu = 2\alpha =\ 2,4,6,8$. With $B_{12},\ B_{34},\ B_{56}$, B_{78} there are up to 8 noncommuting coordinates, a commuting time coordinate, and one (for $p=9$) or zero (for $p=8$) commuting y^I coordinates.

Although it is not indicated above, it is also possible to consider noncommuting timelike coordinates by interchanging a spacelike coordinate with a timelike coordinate in the above configurations and turning on appropriate components of the B-field. But the field theory limit for such configurations exists only with additional conditions (lightlike cases) as described in [14].

Now consider the general constant B-field of rank n, i.e. $B_{1\alpha\,2\beta},\ \alpha,\beta = 1,\cdots,n$. This constant magnetic field B appears in the string Lagrangian in the form

$$B^{1\alpha,2\beta}\left(\partial_\tau X^{1\alpha}\partial_\sigma X^{2\beta} - \partial_\tau X^{2\beta}\partial_\sigma X^{1\alpha}\right). \tag{57}$$

Under the orientifold conditions it does not change sign, because there are two sign changes applied on it, one from reflecting the field (due to $\sigma \leftrightarrow (\pi-\sigma)$ on the string) and one

from changing the orientation of the $X^{2\alpha}$ coordinates. This B-field will lead to the usual noncommutativity in the D-brane coordinates $x^{1\alpha}, x^{2\alpha}$. This explains why, in the algebraic approach of the previous section, the parameter θ does not change sign under transposition.

Now that we have introduced the proper B-field, let us study the supersymmetry of the D_p+O_q system. It is known that in general in the presence of the B-field the conserved supersymmetry is altered e.g. see [4, 15]. Let, Q_L and Q_R denote the supercharges of corresponding type II theory. Introducing the O-plane and D-branes will reduce the supercharges to a combination of Q_L and Q_R, namely:

$$Q = \epsilon_L Q_L + \epsilon_R Q_R \ , \tag{58}$$

where ϵ's are 16 component killing spinors, and should fulfill certain equations. As an example let us consider the brane system given by Eq.(53). For this brane system, the ϵ's should satisfy

$$\Gamma^{012\cdots p}\left(\frac{1}{\sqrt{1+B^2}} + \Gamma^{12}\frac{B}{\sqrt{1+B^2}}\right)\epsilon_L = \epsilon_R \ , \tag{59}$$

and

$$\Gamma^2\Gamma^{012\cdots p+3}\epsilon_L = \epsilon_R \ , \tag{60}$$

where Γ^μ are ten-dimensional 16×16 Dirac matrices. The system will preserve some supersymmetry if Eqs. (59,60) have simultaneous solutions. Solving the above for ϵ_R we find

$$A\, D(B)\epsilon_L = \epsilon_L \ , \tag{61}$$

where

$$A = (-1)^{p-1}\,\Gamma^2\Gamma^{p+1}\Gamma^{p+2}\Gamma^{p+3} \ , \qquad D(B) = \frac{1}{\sqrt{1+B^2}} + \Gamma^{12}\frac{B}{\sqrt{1+B^2}} \ . \tag{62}$$

Then we note that

$$A^2 = \mathbf{1} \ , \qquad AD(B) = D(-B)A \ , \qquad D(B)\ D(-B) = \mathbf{1} \ ,$$

and therefore

$$\left(AD(B)\right)^2 = \mathbf{1} \ . \tag{63}$$

Also noting that $\mathrm{Tr}(AD(B)) = 0$, we conclude that the matrix $AD(B)$ has 8 eigenvalues equal to $+1$ and hence our brane system preserves 8 supercharges. The above argument can easily be repeated for $n = 2, 3, 4$ cases.

The fact that our brane configuration preserves 8 supersymmetries can also be understood if we note that upon a T-duality in the direction parallel to O_q-plane which also contains the B-field (e.g. X^1 direction for the $n = 1$ case of Eq.(53)), our brane system transforms to a usual D_{p-1}+ O_{p+1} system without any B-field.

In the $\alpha' \to 0$ and $B \to \infty$ limit, while $\alpha' B =$ fixed, we expect to find the low energy effective theory of open strings, which should be $\mathrm{o}_\star(N)$ or $\mathrm{usp}_\star(2N)$ gauge theory in $p+1$ dimensions on the worldvolume of the D_p-brane. To show this and also to find the conditions on different field polarizations, let us study the orientifold projection on the gauge fields (massless open string states) in more detail.

As discussed in [16, 17] the orientifold projection, besides the worldsheet parity also involves an operation on the internal degrees of freedom (Chan-Paton factors in our case). Let, $|\psi,\ ij\rangle$ denote the state of an open string before the projection, where ψ is the string oscillatory (space-time) state and i, j are the Chan-Paton indices. The orientifold operation is

$$|\psi,\ ij\rangle \to \gamma^{ik} |\Omega\gamma\psi,\ lk\rangle (\gamma^{lj})^{-1} \tag{64}$$

where Ω is the world-sheet parity and γ is the representation of O-projection on the group indices. For the polarizations of the gauge field parallel to O-plane $\Omega\gamma\psi$ is $-\psi$ and for those which are transverse to O-plane this is $+\psi$. The above conditions can also be written in terms of the gauge fields:

$$\begin{aligned} A_\mu(y^I, x^{1\alpha}; x^{2\alpha}) \quad &\to \quad -\gamma\ A^t_\mu(y^I, x^{1\alpha}; -x^{2\alpha})\gamma^{-1}, \quad \text{for } \mu \text{ parallel to } y^I,\ x^{1\alpha} \text{ directions} \\ A_\mu(y^I, x^{1\alpha}; x^{2\alpha}) \quad &\to \quad +\gamma\ A^t_\mu(y^I, x^{1\alpha}; -x^{2\alpha})\gamma^{-1}, \quad \text{for } \mu \text{ parallel to } x^{2\alpha}\ . \end{aligned} \tag{65}$$

Notice the different overall signs for different polarizations. This can be understood from the vertex operator for gauge fields that is proportional to ∂X_μ. The polarizations associated with $\partial X_{2\alpha}$ have opposite reflection properties compared to those associated with ∂X_0, $\partial X_{1\alpha}$, ∂X_I. Similar conditions apply to the gauginos and any matter supermultiplets.

The consistency (closure) condition for the O-projection requires that

$$\gamma^{-1}\ \gamma^t = \pm 1\ . \tag{66}$$

So, there are two possible choices, symmetric γ, which is $\gamma = \mathbf{1}$ for compact $\mathrm{o}(N)$, and antisymmetric γ, i.e. $\gamma = iC$ (C is given in Eq.(8)), for $\mathrm{usp}(N)$. These two γ choices define our O-plane: the O_q^--plane corresponds to $\gamma = \mathbf{1}$ and O_q^+-plane corresponds to $\gamma = iC$ [1]. From the gauge theory point of view, in fact $\gamma = \mathbf{1}$ reproduces the $\mathrm{o}_\star(N)$ and $\gamma = iC$ the $\mathrm{usp}_\star(N)$ algebra (for even N). Also we recall that N is the total number of D_p-branes (including their reflections from the O_q-plane).

In particular we note that the $\mathrm{o}_\star(1; n)$ theory is a non-trivial one (up to n noncommuting pairs), and it is obtained if we stick an O_q-plane to a single D_p-brane as in Fig.1.

[1] The usual argument that, an O_p-plane plus D_{p+4}-brane system, leads to $\mathrm{sp}(2N)$ (and not $\mathrm{so}(N)$) [17] is basically true when the system includes also some D_p-branes. This is not the case in our system.

We note that here we have just considered the case with 8 supercharges. The classification of the supersymmetric cases and in particular the 16 SUSY case will be studied in another paper [18].

4 Generalization of the Formalism

It is useful to state the problem we have solved more formally in order to provide a more general mathematical structure that could have applications in other areas of physics. Indeed such structures have already appeared before for classifying higher spin algebras [7, 8, 9].

First recall a few definitions. Let B be some algebra with the (not necessarily associative) product law $\diamond$. A map τ of B onto itself is called automorphism if $\tau(a \diamond b) = \tau(a) \diamond \tau(b)$ (i.e., τ is an isomorphism of the algebra to itself.)

A useful fact is that the subset of elements $a \in B$ satisfying

$$\tau(a) = a \tag{67}$$

spans a subalgebra $B_\tau \subset B$. It is customary in physical applications to use this property to obtain reductions. In particular, applying the boson-fermion automorphism that changes a sign of the fermion fields, one obtains reductions to the bosonic sector. Another example is provided by the operation $\tau(a) = -a^t$ of the Lie algebra $gl(N)$ (t implies transposition). The condition (67) then singles out the orthogonal subalgebra $o(N) \subset gl(N)$. From the star product perspective a collection of automorphisms of the star product algebra is provided with the symplectic rotations of the coordinates

$$\tau(x^\nu) = U^\nu{}_\mu x^\mu\,, \tag{68}$$

with

$$U^\nu{}_\rho U^\mu{}_\sigma \theta^{\rho\sigma} = \theta^{\nu\mu}\,. \tag{69}$$

In particular, one can use

$$\tau(x^\nu) = -x^\nu\,. \tag{70}$$

The subalgebra of the star product algebra singled out by the condition (67) is spanned by the even functions

$$f(-x^\nu) = f(x^\nu)\,. \tag{71}$$

Let B be an algebra over the field of complex numbers. If σ is a semilinear involutive homomorphism, i.e.

$$\sigma(\lambda a) = \bar{\lambda}\sigma(a)\,, \qquad \sigma(a \diamond b) = \sigma(a) \diamond \sigma(b)\,, \qquad \sigma^2 = Id \qquad \forall \lambda \in \mathbf{C}\,, \quad a, b \in B \tag{72}$$

it is called conjugation. The set of elements satisfying

$$\sigma(a) = a \tag{73}$$

forms an algebra B_σ over the field of real numbers, called real form of B. For example, in a basis $\{e_i\}$ with real structure coefficients one can define $\sigma(e_i) = e_i$. This way one singles out, e.g., the associative algebra of real matrices $Mat_N(\mathbf{R})$ out of $Mat_N(\mathbf{C})$. The same way, one can single out $gl_N(\mathbf{R})$ from $gl_N(\mathbf{C})$. However, for the Lie algebras there is another option with $\sigma(a) = -(a)^\dagger$ where dagger is the Hermitian conjugation. The resulting real Lie algebra is $u(N)$.

A linear map ρ of an algebra onto itself is called antiautomorphism if it reverses the order of product factors

$$\rho(a \diamond b) = \rho(b) \diamond \rho(a)\,. \tag{74}$$

One example is provided by the transposition of matrices. Antiautomorphisms of the star product algebra are provided by the operations $t_{1,2}$ (20).

A semilinear map μ,

$$\mu(\lambda a) = \bar{\lambda}\mu(a)\,, \qquad \forall \lambda \in \mathbf{C}\,, \quad a \in B \tag{75}$$

of an algebra onto itself, having the property (74) is called second class antiautomorphism. If $\mu^2 = 1$ we will call μ involution. Examples of an involution are provided by the hermitian conjugation of the matrix algebra and the operation $\dagger$ (11) of the star product algebra.

Let now A be some associative algebra over $\mathbf{C}$ with the product law $f \circ g$. Let l_A be the Lie algebra isomorphic to A as a linear space with the Lie product law defined via commutator

$$[f, g] = f \circ g - g \circ f\,. \tag{76}$$

(For example, for $A = Mat_N(\mathbf{C})$, $l_A = gl_N(\mathbf{C})$). Obviously, any automorphism, conjugation, (any class) antiautomorphism or involution of the associative algebra A acts as the operation of the same type in the Lie algebra l_A. However, since the Lie product law is antisymmetric, automorphisms and antiautomorphisms of Lie algebras differ only by sign. Namely, for any antiautomorphism ρ of a Lie algebra l,

$$\tau_\rho = -\rho \tag{77}$$

is its automorphism. Analogously, an involution μ of a Lie algebra l induces its conjugation

$$\sigma_\mu = -\mu\,. \tag{78}$$

As a result, one can define reductions of a Lie algebra l_A with the help of (67) based both on automorphisms and antiautomorphisms of the associative algebra A. Analogously, one can define real forms of l_A using both conjugations and involutions of A.

If A is the tensor product of two associative algebras, $A = A_1 \otimes A_2$, any two operations of the same type (i.e., first or second class (anti)automorphisms) taken in combination define an operation of the same type of A. We denote such combinations $\tau_1 \otimes \tau_2$, $\rho_1 \otimes \rho_2$, $\sigma_1 \otimes \sigma_2$ and $\mu_1 \otimes \mu_2$. Let us emphasize that it is in general impossible to define a sensible operation on $A_1 \otimes A_2$ as a combination of the operations of different types on A_1 and A_2.

More examples are now in order. First, let A be the algebra of $N \times N$ matrices over the field of complex numbers, i.e., $A = Mat_N(\mathbf{C})$ with elements $a^i{}_j$ $(i, j = 1 \div n)$ and product law

$$(a \circ b)^i{}_j = a^i{}_k b^k{}_j \,. \tag{79}$$

Let η^{ij} be a nondegenerate bilinear form with the inverse η_{ij}, i.e.

$$\eta^{ik}\eta_{kj} = \delta^i_j \,. \tag{80}$$

It is elementary to see that the mapping

$$\rho_\eta(a)^i{}_j = \eta^{ik} a^l{}_k \eta_{lj} \tag{81}$$

is an antiautomorphism of $Mat_N(\mathbf{C})$. If the bilinear form η^{ij} is either symmetric

$$\eta^{ij}_S = \eta^{ji}_S \tag{82}$$

or antisymmetric

$$\eta^{ij}_A = -\eta^{ji}_A \tag{83}$$

the antiautomorphism ρ_η is involutive, i.e. $\rho^2_\eta = Id$. For $A = Mat_N(\mathbf{C})$, $l_A = gl_N(\mathbf{C})$. The subalgebras of gl_N singled out by the conditions (67) with $\tau_S = -\rho_S$ and $\tau_A = -\rho_A$ are $\mathrm{o}(N|\mathbf{C})$ and $\mathrm{sp}(N|\mathbf{C})$, respectively, because the conditions (67) just imply that the form η^{ij} is invariant. Analogously, one can define involutions via nondegenerate hermitian forms. If $\dagger$ is such an involution of $Mat_N(\mathbf{C})$ defined via a positive-definite Hermitian form, then the real form of $gl_N(\mathbf{C})$ defined via (73) with $\sigma = -\dagger$ is spanned by antihermitian matrices, thus being $\mathrm{u}(N)$.

Let now A be the star product algebra. From the defining relations (4) it follows that the definition of an involution

$$(x^\nu)^\dagger = x^\nu \tag{84}$$

is consistent and therefore extends to the whole star product algebra. In the particular basis associated with the Weyl (i.e. totally symmetric) ordering, in which the star product has

the Moyal form (2), the reordering of the operators has no effect and, therefore, the formula (11) is true. Analogously, any linear map

$$\rho(x^\nu) = U(x)^\nu = U^\nu{}_\mu x^\mu \tag{85}$$

such that

$$U^\nu{}_\eta U^\mu{}_\kappa \theta^{\eta\kappa} = -\theta^{\nu\mu} \tag{86}$$

induces an antiautomorphism ρ_U of the star product algebra. Again, because of using the Weyl ordering, its action on a general element is simply

$$\rho_U(f(x)) = f(U(x))\,. \tag{87}$$

The action of the antiautomorphism on the matrix part was defined as in the example (81).

The examples of $\mathrm{o}_\star$ and $\mathrm{usp}_\star$ algebras given in this paper are obtained from the application of this general scheme to the Lie algebra l_A with $A = Mat_N \otimes \mathcal{A}$. For the particular case of only two noncommutative coordinates, the map (86) was taken in the one of the two forms

$$U_1(x^1) = x^1\,, \qquad U_1(x^2) = -x^2\,, \tag{88}$$

or

$$U_2(x^1) = x^2\,, \qquad U_2(x^2) = x^1\,. \tag{89}$$

The fact that the reduction was defined with the help of an automorphism of the algebra implies that it is consistent with the definition of the noncommutative Yang-Mills curvatures. Let us consider for definiteness the case when all coordinates are non-commutative, i.e. $\theta^{\mu\nu}$ is nondegenerate, thus having inverse $\theta_{\mu\nu}$. Let us now introduce the 1-form gauge potential

$$\mathcal{A} = dx^\mu \mathcal{A}_\mu = dx^\mu(-i\theta_{\mu\nu}x^\nu I + A_\mu\,(x))\,. \tag{90}$$

where the star commutator of $-i\theta_{\mu\nu}x^\nu$ with any function $f\,(x)$ is the derivative $\partial_\mu f$. The term $dx^\mu\,(-i\theta_{\mu\nu}x^\nu I)$ can be treated as the vacuum value of the potential.We obtain

$$\mathcal{A} \wedge *\mathcal{A} = dx^\mu \wedge dx^\nu(-i\theta_{\mu\nu}I + G_{\mu\nu})\,, \tag{91}$$

where $G_{\mu\nu}$ is the field strength (17) with the matrix indices implicit.

If τ is some automorphism of the Lie algebra built through commutators in $Mat_N \otimes \mathcal{A}$, this implies that

$$\tau(\mathcal{A} \wedge *\mathcal{A}) = \tau(\mathcal{A}) \wedge *\tau(\mathcal{A})\,. \tag{92}$$

To make it possible to truncate the system by imposing the condition

$$\tau(\mathcal{A}) = \mathcal{A}, \tag{93}$$

it is necessary to insure that the vacuum value of the potential is invariant. To this end one has to extend the action of $\tau \to \tau'$ to the wedge algebra by requiring

$$\tau'(dx^\mu\,(-i\theta_{\mu\nu}x^\nu I)) = dx^\mu\,(-i\theta_{\mu\nu}x^\nu I)\,, \tag{94}$$

which is possible for any τ that acts linearly on $(\theta_{\mu\nu}x^\nu)$

$$\tau'(\theta_{\mu\nu}x^\nu) = \tau(\theta_{\mu\nu}x^\nu) = V_\mu{}^\nu\left(\theta_{\nu\lambda}x^\lambda\right) \tag{95}$$

by defining

$$\tau'(dx^\mu) = dx^\lambda\left(V^{-1}\right)_\lambda{}^\mu\,. \tag{96}$$

Simultaneously, one has to redefine the action of τ on the potential

$$\tau'(\mathcal{A}_\mu) = V_\mu{}^\nu\tau(\mathcal{A}_\nu)\,. \tag{97}$$

As a result, the potentials can be consistently restricted by the condition

$$\tau'(\mathcal{A}) = \mathcal{A}, \tag{98}$$

which is consistent with the field strength satisfying the similar condition

$$\tau'(\mathcal{A}*\wedge\mathcal{A}) = \mathcal{A}*\wedge\mathcal{A} \tag{99}$$

as a consequence of the fact that τ' is an automorphism. The additional signs in the transformation laws of the potentials and the field strengths discussed in section 2 are just the particular realizations of the definition (97).

In the classical limit with commuting coordinates the associative algebra of functions is commutative so that there is no difference between its antiautomorphisms and automorphisms. This allows one to use the identical (anti)automorphism of the algebra of functions in the standard construction of the usual (i.e., commutative non-Abelian) Yang-Mills theory. In the non-commutative case this is not allowed any longer. As a result, the classical limit of our $o_\star$ and $usp_\star$ noncommutative Yang-Mills theories is different from the usual Yang Mills theory, because its matrix content is dependent on the oddness of the Yang-Mills potentials as in (31), as discussed at the end of section 2.2.

Let us note that in [7, 8] where similar technics were originally applied to the tensor product of the star product algebra with matrix algebras, relevant to the problem of higher spin gauge fields [9], it was discussed in a more general framework of Lie superalgebras rather than Lie algebras. For more detail on the relationships between (semi)linear (anti)automorphisms associative algebras and real forms and reductions of the associated Lie superalgebras built via (anti)commutators we therefore refer the reader to the first reference in [7].

5 Outlook

In this work we have studied the formulation of noncommutative $o_\star(N)$ and $usp_\star(2N)$ algebras and the corresponding gauge theories. Our method is based on the specific realization of an antiautomorphism ρ that changes the order of the functions in the star products, Eq. (10). We showed that the map ρ can be obtained through an operation which acts on the noncommutative coordinates, the t_1 or t_2 operations. In this way we can relax the θ dependence of functions which were assumed in [4] for the representation of this map. The $t_{1,2}$ transformations do not change θ.

Then we discussed how the combination of the t_1 (or t_2) with the usual matrix transposition, the T operation, provides the proper "transposition" for the general star-matrix algebras. Particularly starting with the $u_\star(N)$ algebra, and restricting it to the anti-symmetric elements under T operation, leads to $o_\star(N)$ as a subalgebra. Similarly one can construct $usp_\star(2N)$ as a subalgebra of $u_\star\,(2N)$. In order to formulate $o_\star(N)$ field theories, including scalar, fermion and vector fields, we studied the proper representations and showed that for the gauge fields, the T operation also imposes some conditions on the polarizations which are not the same in every direction.

We noted that in the $\theta \to 0$ limit the noncommutative Yang-Mills theory does not necessarily reduce to the standard Yang-Mills theory if there is no reduction in the number of dimensions. This is because of the twisted symmetry properties of the different polarizations of the gauge fields under the transposition T. However, one may consider several different types of $\theta \to 0$ limits in which various quantities are held fixed. In particular one may consider the limit in which the $x_{2\alpha}$ coordinates are dimensionally reduced. Then the resulting limiting theory is a Yang Mills theory in lower dimensions with additional scalars in various representations.

We also discussed the brane configurations with 8 supercharges which lead to the $o_\star(N)$ and $usp_\star(2N)$ gauge theories. We discussed how exactly the same conditions and requirements that we considered from an algebraic point of view also follows from string theory.

The discussion of other D_p+O_q configurations in the presence of the B field with other amounts of supersymmetry, and the structure of the corresponding supersymmetric $o_\star(N)$ or $usp_\star\,(2N)$ gauge theories are left for a future paper [18].

Let us mention some possible areas where our results could be useful. One potential application of the $o_\star(N)$ group is in the context of noncommutative gravity. Previously, because a noncommutative version of so(N) was lacking, noncommutative gravity was attempted by gauging $u_\star(3,1)$ [19]. Among other challenges of this approach, $u_\star(3,1)$ has the disadvantage that the gravity field, the metric, becomes a complex field. However, it seems

more plausible to attempt a construction of noncommutative gravity by gauging $\mathrm{o}_\star(3,1)$, which is an analytic continuation of the $\mathrm{o}_\star(4)$ we discussed here.

Another interesting open problem one could address for the 16 SUSY case is the corresponding noncommutative open string (NCOS) theory [20]. Usually the NCOS appears in noncommutative space-times, in the critical noncommutativity limit (when the noncommutativity scale and the string scale become identical.) In our case, we expect that the same critical limit exists and will lead to unoriented NCOS [21].

It is known that the $D = 4,\ N = 2$ theories can be studied through the holomorphic Seiberg-Witten curve [22]. On the other hand it has been shown that the same curve can be understood more intuitively through brane configurations involving type IIA $NS5$-branes [23]. For the $\mathrm{u}_\star(N)$ theories the corresponding brane configuration have been discussed [15]. Since adding the $NC5$-brane and the orientifolds do not break any further supersymmetries [24] we expect that the $\mathrm{o}_\star(N)$ theories can be studied through the "curve" obtained from our brane configurations (of course after adding the needed $NS5$-branes).

As a first extension of our work one may look for the superalgebraic generalization of the $\mathrm{o}_\star(N)$ and $\mathrm{usp}_\star(2N)$ algebras.

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